Quiz 7 – Solutions

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1. Complete[*] the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function $T : V \to W$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars a, b,

   $$T(a\mathbf{u} + b\mathbf{v}) \;=\; a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$

   Equivalently: (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and (ii) $T(a\mathbf{u}) = a\,T(\mathbf{u})$.

   (b) A *vector* is ...

   Solution: An element of a vector space. Concretely, if V is a vector space over a field $\mathbb{F}$, any $v \in V$ is called a vector. (For $V = \mathbb{R}^n$, a vector is an ordered n-tuple $[x_1, \ldots, x_n]^T$ with componentwise addition and scalar multiplication.)

2. Suppose $n \in \mathbb{Z}_{>0}$. Recall that $\mathcal{P}_n$ denotes the polynomials of degree less than or equal to n. Show

   VS-3: There is an element $\vec{0} \in \mathcal{P}_n$ such that $p(x) + \vec{0} = p(x)$ for all $p(x) \in \mathcal{P}_n$.

   Solution: Take $\vec{0}$ to be the *zero polynomial* $0(x) \equiv 0$. Since $\deg 0 \le n$, we have $0 \in \mathcal{P}_n$. For any $p \in \mathcal{P}_n$ and any x,

   $$\big(p + 0\big)(x) = p(x) + 0(x) = p(x),$$

   so $p + 0 = p$. Thus 0 is the additive identity in $\mathcal{P}_n$, as required.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Suppose $A, B \in \mathbb{R}^{2\times 2}$. We have $AB = BA$.

   Solution: FALSE. In general matrices do not commute. For instance,

   $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[*]For full credit, please write out fully what you mean instead of using shorthand phrases.

Then

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad BA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and $AB \neq BA$.

(b) Suppose $\mathbb{R}^n \xrightarrow{T_M} \mathbb{R}^m$ and $\mathbb{R}^m \xrightarrow{T_N} \mathbb{R}^p$ are linear transformations with standard matrices M and N respectively. The composition $T_N \circ T_M$ is a linear transformation with standard matrix NM.

Solution: TRUE. For any $\mathbf{x} \in \mathbb{R}^n$,

$$(T_N \circ T_M)(\mathbf{x}) = T_N\big(T_M(\mathbf{x})\big) = T_N(M\mathbf{x}) = N(M\mathbf{x}) = (NM)\mathbf{x}.$$

Thus the standard matrix of the composition is NM (note the order), and compositions of linear maps are linear.